THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5220 Complex Analysis and its Applications 2016-2017
Suggested Solution to Assignment 7

Update at 25/4/2017:

- Some typos in Question 3 and 4 have been fixed.

1 (a) Let $f(z) = \dfrac{3z^2}{(z^2+1)(z^2+4)}$. For $R > 4$, consider the positively oriented contour $C(R) = [-R, R] \cup C^+(R)$, where $C^+(R) = \{Re^{i\theta} \mid \theta \in [0, \pi]\}$. By residue theorem, we have

$$\begin{aligned}\int_{-R}^{R} f(x)dx + \int_{C^+(R)} f(z)dz =& 2\pi i \left(\operatorname{Res}(f, i) + \operatorname{Res}(f(z), 2i)\right) \\ =& 2\pi i \left(\frac{3(i)^2}{(i+i)(i^2+4)} + \frac{3(2i)^2}{((2i)^2+1)(2i+2i)} \right) \\ =& \pi\end{aligned}$$

Furthermore, by triangle inequality,

$$\left| \int_{C^+(R)} \frac{3z^2}{(z^2+1)(z^2+4)} dz \right| \leq \pi R \times \frac{3R^2}{(R^2-1)(R^2-4)} \xrightarrow{R\to\infty} 0$$

As a result, we have

$$\int_0^{\infty} f(x)dx = \frac{1}{2} \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2}$$

(b) Let $f(z) = \dfrac{1}{2z^2+2z+1}$. For $R > 2$, Consider the positively oriented contour $C(R) = [-R, R] \cup C^+(R)$, where $C^+(R) = \{Re^{i\theta} \mid \theta \in [0, \pi]\}$. By residue theorem, we have

$$\begin{aligned}\int_{-R}^{R} f(x)dx + \int_{C^+(R)} f(z)dz =& 2\pi i \operatorname{Res}(f, \frac{-1+i}{2}) \\ =& 2\pi i \frac{1}{2(\frac{-1+i}{2} - \frac{-1-i}{2})} \\ =& \pi\end{aligned}$$

Furthermore, by triangle inequality,

$$\left| \int_{C^+(R)} \frac{1}{2z^2+2z+1} dz \right| \leq \pi R \times \frac{1}{2R^2-2R-1} \xrightarrow{R\to\infty} 0$$

As a result, we have

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \pi$$

(c) Let $f(z) = \dfrac{1}{z^2+4}$. For $R > 4$, consider the positively oriented contour $C(R) = [-R, R] \cup C^+(R)$, where $C^+(R) = \{Re^{i\theta} \mid \theta \in [0, \pi]\}$. By residue theorem, we have

$$\begin{aligned}\int_{-R}^{R} f(x)e^{iax}dx + \int_{C^+(R)} f(z)e^{iaz}dz =& 2\pi i \operatorname{Res}(fe^{iaz}, 2i) \\ =& 2\pi i \frac{e^{ia(2i)}}{2i+2i} \\ =& \frac{\pi e^{-2a}}{2}\end{aligned}$$

Furthermore, by Jordan lemma, since $|f(x)| \leq \dfrac{1}{R^2-4} \xrightarrow{R\to\infty} 0$, we have

$$\int_{C^+(R)} f(z)e^{iaz}dz \xrightarrow{R\to\infty} 0$$

As a result, we have

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = \frac{\pi e^{-2a}}{2},$$

which implies

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+4}dx = \frac{\pi e^{-2a}}{2}$$

As a result,

$$\int_{0}^{\infty} \frac{\cos ax}{x^2+4}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+4}dx = \frac{\pi e^{-2a}}{4}$$

(d) Let $f(z) = \dfrac{z}{2z^2+2z+1}$. For $R > 4$, Consider the positively oriented contour $C(R) = [-R, R] \cup C^+(R)$, where $C^+(R) = \{Re^{i\theta} \mid \theta \in [0, \pi]\}$. By residue theorem, we have

$$\begin{aligned}\int_{-R}^{R} f(x)e^{i2x}dx + \int_{C^+(R)} f(z)e^{i2z}dz =& 2\pi i \operatorname{Res}(fe^{i2z}, \frac{-1+i}{2}) \\ =& 2\pi i \frac{(\frac{-1+i}{2})e^{i2(\frac{-1+i}{2})}}{2(\frac{-1+i}{2} - \frac{-1-i}{2})} \\ =& \pi e^{-1-i}(\frac{-1+i}{2})\end{aligned}$$

Furthermore, by Jordan lemma, since $|f(x)| \leq \dfrac{R}{2R^2-2R-1} \xrightarrow{R\to\infty} 0$, we have

$$\int_{C^+(R)} f(z)e^{i2z}dz \xrightarrow{R\to\infty} 0$$

As a result, we have

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)e^{i2x}dx = \pi e^{-1-i}(\frac{-1+i}{2}),$$

which implies

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x\sin 2x}{2x^2+2x+1}dx = \frac{\pi e^{-1}}{2}(\cos 1 + \sin 1)$$

2 Let $f(z) = \frac{z}{z^2-1}$. For $R > 4$, consider the positively oriented contour $C(R) = [-R, -1-\epsilon] \cup C^+(-1,\epsilon) \cup [-1+\epsilon, 1-\epsilon] \cup C^+(1,\epsilon) \cup [1+\epsilon, R] \cup C^+(R)$, where $C^+(R) = \{Re^{i\theta} \mid \theta \in [0,\pi]\}$ and $C^+(\pm 1, \epsilon) = \{\epsilon e^{i\theta} \pm 1 \mid \theta \in [0,\pi]\}$. By residue theorem, since $f(z)e^{i2z}$ is analytic inside $C(R)$, we have

$$\int_{C(R)} f(z)e^{i2z}dz = 0$$

By Jordan lemma, since $|f(z)| \leq \frac{R}{R^2-1} \xrightarrow{R\to\infty} 0$, we have

$$\int_{C^+(R)} f(z)e^{i2z}dz \xrightarrow{R\to\infty} 0$$

Moreover,

$$\int_{C^+(1,\epsilon)} f(z)e^{i2z}dz + \int_{C^+(-1,\epsilon)} f(z)e^{i2z}dz = \pi i(\text{Res}(f(z)e^{i2z}, 1) + \text{Res}(f(z)e^{i2z}, -1)) = \frac{\pi i(e^{2i}+e^{-2i})}{2}$$

Therefore, we have

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2+1}dx = \text{Im}(\frac{\pi i(e^{2i}+e^{-2i})}{2}) = \pi\cos 2$$

3 Let $f(z) = \frac{1}{z^3}$. For $R > 4$, consider the positively oriented contour $C(R) = [-R, -\epsilon] \cup C^+(\epsilon) \cup [\epsilon, R] \cup C^+(R)$, where $C^+(R) = \{Re^{i\theta} \mid \theta \in [0,\pi]\}$ and $C^+(\epsilon) = \{\epsilon e^{i\theta} \mid \theta \in [0,\pi]\}$. By residue theorem, since $f(z)(\frac{3}{4}e^{iz} - \frac{1}{4}e^{i3z} - \frac{1}{2})$ is analytic inside $C(R)$, we have

$$\int_{C(R)} f(z)(\frac{3}{4}e^{iz} - \frac{1}{4}e^{i3z} - \frac{1}{2})dz = 0$$

Note that $|\int_{C^+(R)} \frac{1}{2}f(z)dz| \leq \pi R \times \frac{1}{2R^3} \xrightarrow{R\to\infty} 0$. Furthermore, by Jordan lemma, since $|f(z)| \leq \frac{1}{R^3} \xrightarrow{R\to\infty} 0$, we have

$$\int_{C^+(R)} f(z)e^{iz}dz \xrightarrow{R\to\infty} 0 \text{ and } \int_{C^+(R)} f(z)e^{i3z}dz \xrightarrow{R\to\infty} 0$$

Moreover,

$$\begin{aligned}
&\int_{C^+(0,\epsilon)} f(z)(\frac{3}{4}e^{iz} - \frac{1}{4}e^{i3z} - \frac{1}{2})dz \\
=&\pi i \,\text{Res}\left(f(z)(\frac{3}{4}e^{iz} - \frac{1}{4}e^{i3z} - \frac{1}{2}), 0\right) \\
=&\pi i \,\text{Res}\left(\frac{1}{z^3}\left(\frac{3}{4}(1+(iz)+\frac{(iz)^2}{2}+\dots) - \frac{1}{4}(1+(3iz)+\frac{(3iz)^2}{2}+\dots) - \frac{1}{2}\right), 0\right) \\
=&\frac{3\pi i}{4}
\end{aligned}$$

Therefore, we have

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}dx = \text{Im}(\frac{3\pi i}{4}) = \frac{3\pi}{4}$$

4 Consider the function $f(z) = \dfrac{\sqrt{z}}{z^2+1}$ with the branch cut along positive x-axis. Consider the contour $C = C_R + L_1 + C_\epsilon + L_2$, where $C_R = \{Re^{i\theta} \mid \theta \in [0, 2\pi]\}$, $L_1 = \{(\epsilon - R)t + R \mid t \in [0,1]\}$, $C_\epsilon = \{\epsilon e^{i(2\pi-\theta)} \mid \theta \in [0, 2\pi]\}$ and $L_2 = \{(R-\epsilon)t + \epsilon \mid t \in [0,1]\}$.

On L_1, $\log z = \ln r + 2\pi i$. On L_2, $\log z = \ln r$. Therefore,

$$\int_{L_1} f(z)dz = \int_R^\epsilon \frac{e^{\frac{1}{2}(\ln r + 2\pi i)}}{r^2+1} dr = \int_\epsilon^R \frac{\sqrt{r}}{r^2+1} dr \qquad \text{and}$$

$$\int_{L_2} f(z)dz = \int_\epsilon^R \frac{e^{\frac{1}{2}(\ln r)}}{r^2+1} dr = \int_\epsilon^R \frac{\sqrt{r}}{r^2+1} dr = \int_{L_1} f(z)dz$$

On the other hand,

$$\left| \int_{C_\epsilon} f(z)dz \right| \leq 2\pi\epsilon \frac{\sqrt{\epsilon}}{1-\epsilon^2} \xrightarrow{\epsilon \to 0} 0 \text{ and } \left| \int_{C_R} f(z)dz \right| \leq 2\pi R \frac{\sqrt{R}}{R^2-1} \xrightarrow{R \to \infty} 0$$

As a result,

$$\begin{aligned}
2\int_0^\infty f(z)dz =& 2\pi i \left(\mathrm{Res}(f, i) + \mathrm{Res}(f, -i)\right) \\
=& 2\pi i \left(\frac{\sqrt{i}}{i+i} + \frac{\sqrt{-i}}{-i-i} \right) \\
=& 2\pi i \left(\frac{e^{\frac{1}{2}(\ln(1) + i(\frac{\pi}{2}))}}{2i} + \frac{e^{\frac{1}{2}(\ln(1) + i(\frac{3\pi}{2}))}}{-2i} \right) \\
=& \sqrt{2}\pi
\end{aligned}$$

Hence

$$\int_0^\infty f(z)dz = \frac{\sqrt{2}\pi}{2}$$